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Ordinary differential systems describing hysteresis phenomena and numerical simulation

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1 Introduction

In this paper we deal with a nonlinear ordinary differential system which describes hysteresis input-output relations. Let us consider a system of the following form:

$$aw' + bu' + \partial I_u(w) \ni F(u, w) \text{ in } (0, \infty), \tag{1.1}$$

$$cw' + du' = h(u, w) \text{ in } (0, \infty), \tag{1.2}$$

subject to the initial conditions:

$$u(0) = u_0,\ w(0) = w_0, \tag{1.3}$$

where $a > 0, b < 0, c > 0, d > 0$ are given constants, $F, h : R \times R \to R$ are Lipschitz continuous functions, $f_*, f^* : R \to R$ are non-decreasing Lipschitz continuous functions with $f_* \leq f^*$, $I_u(\cdot)$ is the indicator function of the closed interval $[f_*(u), f^*(u)]$, and $\partial I_u(\cdot)$ is its subdifferential defined by

$$\partial I_u(w) = \begin{cases} \emptyset & \text{for } w > f^*(u) \text{ or } w < f_*(u), \\ [0, +\infty) & \text{for } w = f^*(u) > f_*(u), \\ \{0\} & \text{for } f_*(u) < w < f^*(u), \\ (-\infty, 0] & \text{for } w = f_*(u) < f^*(u), \\ (-\infty, +\infty) & \text{for } w = f_*(u) = f^*(u). \end{cases} \tag{1.4}$$

Equation (1.1) describes a lot of input-output relations $u \to w$ which are physically relevant. For example, when $b = 0$ (resp. -1), $a = 1$ and $F \equiv 0$, the relation between $w(t)$ and $u(t)$ is called a play (resp. stop) operator. These operators are typical examples of hysteresis input-output relations, and are used to present various phase transition effects. Moreover, in the case when $a = 1, b = 0, c = 1, d = 1, F \equiv 0, h \equiv 0$, the system was studied by Visintin [5]. In the general case when $a = a(u, w), b = b(u, w), c = c(u, w), d = d(u, w)$ are functions of u, w with $a(u, w) > 0, c(u, w) > 0, d(u, w) > 0$ and $a(u, w)d(u, w) - b(u, w)c(u, w) > 0$, the existence and uniqueness results of the system were obtained in [2].

Our main objective of this paper is to study the large time behaviour of solutions of our system. The behaviour of solutions of (1.1),(1.2) depends on the coefficients a, b, c, d and the functions F, h. Under some conditions on a, b, c, d, F, h and f_*, f^*, we investigate the precise behaviour of orbits of solutions of our system. At the same time, we give some numerical experiments for the connection with the behaviour of the orbits.

2 Preliminaries and main results

In this section, we mention the precise assumptions on the coefficients a, b, c, d and the functions F, h, f_*, f^*, and a theoretical result on the behaviour of orbits of solutions of our system. Now we make the following assumptions:

(A1) $F := \alpha u + \beta w,\ h := \gamma u + \delta w,\ \alpha, \beta, \gamma, \delta \in R$
and $c\alpha - a\gamma = d\beta - b\delta = 0,\ d\alpha - b\gamma > 0,\ c\beta - a\delta > 0$.

(A2) Functions f_*, f^* are non-decreasing Lipschitz continuous functions of C^2-class such that $f_*(u) \le f^*(u)$ for all $u \in R$, and there are constants $f^\infty > 0$, $f_\infty < 0$ and $\kappa^* > 0, \kappa_* < 0$ such that
$f_*(u) = f^*(u) \equiv f^\infty$ for all sufficiently large $u > 0$,
$f_*(0) < 0 < f^*(0)$,
$f_*(u) = f^*(u) \equiv f_\infty$ for all sufficiently small $u < 0$,
$f_*(u) = f^*(u)$ for $u \in (-\infty, \kappa_*] \cup [\kappa^*, +\infty)$.

(A3) The number of connected components of the sets
$\{u \in R | (a\delta - c\beta) f_*(u) f_*'(u) - (d\alpha - b\gamma)u = 0\}$ and
$\{u \in R | (a\delta - c\beta) f^*(u) f^{*\prime}(u) - (d\alpha - b\gamma)u = 0\}$ is finite.

Assumption (A1) means that if there is no subdifferential $\partial I_u(w)$ in our system, then the orbits of solutions are anticlockwise ellipse for all initial data (especially the orbits of solutions are anticlockwise circles when $d\alpha - b\gamma = c\beta - a\delta > 0$ hold). Assumptions (A2), (A3) are concerned with the geometry of the two curves $w = f^*$ and $w = f_*$. Especially, assumption (A3) implies that the curves $w = f_*(u)$ and $w = f^*(u)$ have a finite number of circles with center $(0,0)$ which are tangential to the curves $w = f_*(u)$ or $w = f^*(u)$.

Under these assumptions, we give the definition of a solution of our system.

Definition 2.1 *A pair of functions $\{w, u\}$ is called a solution of the system (1.1), (1.2), and (1.3) if the following (1)-(4) are satisfied:*

(1) $w, u \in W^{1,2}(0,T)$ *for any finite* $T > 0$,

(2) $aw' + bu' + \partial I_u(w) \ni \alpha u + \beta w$ *a.e. on* $(0,\infty)$,

(3) $cw' + du' = \gamma u + \delta w$ *on* $(0,\infty)$,

(4) $u(0) = u_0,\ w(0) = w_0$.

The following theorem holds true.

Theorem 2.1 *Under these assumptions, the system (1.1)-(1.3) possesses one and only one solution.*

This theorem guarantees the existence and uniqueness of solutions and it is a special case of [2; Theorem 2.4].

The precise behaviour of solutions of our system is given in the following theorem.

Theorem 2.2 *Suppose that assumptions (A1),(A2) and (A3) are satisfied. Let* $\mathcal{S} = \{(u,w) \in R^2 | f_*(u) \leq w \leq f^*(u)\}$, *and denote by* $\{u,w\}$ *the solution of our system with initial values* u_0, w_0. *Then* $\mathcal{S}$ *is divided into the following three subsets* $\mathcal{S}_1, \mathcal{S}_2$ *and* $\mathcal{S}_3$, *i.e.* $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$, *such that*

(i) if $(u_0, w_0) \in \mathcal{S}_1$, *then* $(u(t), w(t))$ *reaches a periodic ellipse around the origin in a finite time;*

(ii) if $(u_0, w_0) \in \mathcal{S}_2$, *then* $(u(t), w(t))$ *converges (as* $t \to +\infty$*) to a stationary point* (u_∞, w_∞) *which satisfies*

$$\begin{cases} \partial I_{u_\infty}(w_\infty) \ni \alpha u_\infty + \beta w_\infty \\ \gamma u_\infty + \delta w_\infty = 0; \end{cases}$$

(iii) if $(u_0, w_0) \in \mathcal{S}_3$, *then* $(u(t), w(t))$ *diverges to* $(+\infty, f^\infty)$ *or to* $(-\infty, f_\infty)$ *as* $t \to +\infty$.

Moreover, the sets $\mathcal{S}_1, \mathcal{S}_2$ *and* $\mathcal{S}_3$ *are determined by the geometries of the curves* $w = f_*(u)$, $w = f^*(u)$ *and the line* $\gamma u + \delta w = 0$ *and their expressions are given in the next section.*

In order to prove Theorem 2.2, we prepare the following section.

3 Subsets $\mathcal{S}_i$ $(i = 1, 2, 3)$

In this section, we consider how to describe the subsets $\mathcal{S}_i (i = 1, 2, 3)$ of $\mathcal{S}$ on (u, w) plane. Now we use the following notations:

$$\Gamma^* := \{(u, w) | w = f^*(u)\},\ \Gamma_* := \{(u, w) | w = f_*(u)\},$$
$$\mathcal{B}(u, w) := \{(d\alpha - b\gamma)u^2 + (c\beta - a\delta)w^2\}^{\frac{1}{2}},\ l := \{(u, w) \in R^2 | \gamma u + \delta w = 0\},$$
$$\Gamma^*(l) := \{(u, w) \in \Gamma^* \cap l | u > 0\},\ \Gamma_*(l) := \{(u, w) \in \Gamma_* \cap l | u < 0\},$$
$$r_0^* := \min\{\mathcal{B}(u, w) | (u, w) \in \Gamma^*\},\ u^* := \min\{u | (u, w) \in \Gamma^*, \mathcal{B}(u, w) = r_0^*\},$$
$$r_{0*} := \min\{\mathcal{B}(u, w) | (u, w) \in \Gamma_*\},\ u_* := \max\{u | (u, w) \in \Gamma_*, \mathcal{B}(u, w) = r_{0*}\},$$
$$r_1^* := \min\{\mathcal{B}(u, w) | (u, w) \in \Gamma^*(l)\},\ R_1^* := \max\{\mathcal{B}(u, w) | (u, w) \in \Gamma^*(l)\},$$
$$r_{1*} := \min\{\mathcal{B}(u, w) | (u, w) \in \Gamma_*(l)\},\ R_{1*} := \max\{\mathcal{B}(u, w) | (u, w) \in \Gamma_*(l)\},$$
$$A^+ := \left\{ (u, w) \left| \begin{array}{l} u_* w - f_*(u_*) u \geq 0 \text{ if } u \geq 0 \\ u^* w - f^*(u^*) u < 0 \text{ if } u < 0 \end{array} \right. \right\},$$
$$A^- := \left\{ (u, w) \left| \begin{array}{l} u_* w - f_*(u_*) u < 0 \text{ if } u > 0 \\ u^* w - f^*(u^*) u \geq 0 \text{ if } u \leq 0 \end{array} \right. \right\},$$
$$\mathcal{S}_0 := \{(u, w) \in \mathcal{S} | \mathcal{B}(u, w) \leq r_0\} \text{ with } r_0 := \min\{r_0^*, r_{0*}\}.$$

By our assumptions, we have

$$r_0^* < r_1^* \leq R_1^* \text{ and } r_{0*} < r_{1*} \leq R_{1*}.$$

As to the relationships of $r_0^*, r_1^*, R_1^*, r_{0*}, r_{1*}$ and R_{1*} there are the following 6 cases to be considered:

$$\begin{array}{llll} (1) & r_{0*} \leq r_0^* < r_{1*} \leq R_{1*} & (2) & r_{0*} < r_{1*} \leq r_0^* < R_{1*} \\ (3) & r_{0*} < r_{1*} \leq R_{1*} < r_0^* & (4) & r_0^* \leq r_{0*} < r_1^* \leq R_1^* \\ (5) & r_0^* < r_1^* \leq r_{0*} \leq R_1^* & (6) & r_0^* < r_1^* \leq R_1^* < r_{0*} \end{array}$$

In the case of (1) we define

$$\mathcal{S}_1 := \mathcal{S}_0 \cup \mathcal{S}_1^+ \cup \mathcal{S}_1^-, \tag{3.1}$$

where

$$\mathcal{S}_1^+ := \{(u, w) \in \mathcal{S} \cap A^+ | r_{0*} < \mathcal{B}(u, w) < r_1^*\}, \tag{3.2}$$

$$\mathcal{S}_1^- := \{(u, w) \in \mathcal{S} \cap A^- | r_{0*} < \mathcal{B}(u, w) < r_{1*}\}; \tag{3.3}$$

$$\mathcal{S}_2 := \mathcal{S}_2^+ \cup \mathcal{S}_2^-, \tag{3.4}$$

where

$$\mathcal{S}_2^+ := \{(u, w) \in \mathcal{S} \cap A^+ | r_1^* \leq \mathcal{B}(u, w) \leq R_1^*\}, \tag{3.5}$$

$$\mathcal{S}_2^- := \{(u,w) \in \mathcal{S} \cap A^- | r_{1*} \le \mathcal{B}(u,w) \le R_{1*}\}; \tag{3.6}$$

$$\mathcal{S}_3 := \mathcal{S}_3^+ \cup \mathcal{S}_3^-, \tag{3.7}$$

where

$$\mathcal{S}_3^+ := \{(u,w) \in \mathcal{S} \cap A^+ | R_1^* < \mathcal{B}(u,w)\}, \tag{3.8}$$

$$\mathcal{S}_3^- := \{(u,w) \in \mathcal{S} \cap A^- | R_{1*} < \mathcal{B}(u,w)\}. \tag{3.9}$$

In the case of (2) we define

$$\mathcal{S}_1 := \mathcal{S}_0 \cup \mathcal{S}_1^0, \tag{3.10}$$

where

$$\mathcal{S}_1^0 := \{(u,w) \in \mathcal{S} | r_{0*} < \mathcal{B}(u,w) < r_{1*}\}; \tag{3.11}$$

$$\mathcal{S}_2 := \mathcal{S}_2^+ \cup \mathcal{S}_2^-, \tag{3.12}$$

where

$$\mathcal{S}_2^+ := \{(u,w) \in \mathcal{S} \cap A^+ | r_1^* \le \mathcal{B}(u,w) \le R_1^*\}, \tag{3.13}$$

$$\begin{aligned} \mathcal{S}_2^- : \quad = \quad & \{(u,w) \in \mathcal{S} \cap A^+ | r_{1*} \le \mathcal{B}(u,w) < r_1^*\} \\ & \cup \{(u,w) \in \mathcal{S} \cap A^- | r_{1*} \le \mathcal{B}(u,w) \le R_{1*}\}; \end{aligned} \tag{3.14}$$

$$\mathcal{S}_3 := \mathcal{S}_3^+ \cup \mathcal{S}_3^-, \tag{3.15}$$

where

$$\mathcal{S}_3^+ := \{(u,w) \in \mathcal{S} \cap A^+ | R_1^* < \mathcal{B}(u,w)\}, \tag{3.16}$$

$$\mathcal{S}_3^- := \{(u,w) \in \mathcal{S} \cap A^- | R_{1*} < \mathcal{B}(u,w)\}. \tag{3.17}$$

In the case of (3) we define

$$\mathcal{S}_1 := \mathcal{S}_0 \cup \mathcal{S}_1^0, \tag{3.18}$$

where

$$\mathcal{S}_1^0 := \{(u,w) \in \mathcal{S} | r_{0*} < \mathcal{B}(u,w) < r_{1*}\}; \tag{3.19}$$

$$\mathcal{S}_2 := \mathcal{S}_2^+ \cup \mathcal{S}_2^-, \tag{3.20}$$

where

$$\mathcal{S}_2^+ := \{(u,w) \in \mathcal{S} \cap A^+ | r_1^* \le \mathcal{B}(u,w) \le R_1^*\}, \tag{3.21}$$

$$\mathcal{S}_2^- := \{(u,w) \in \mathcal{S} | r_{1*} \le \mathcal{B}(u,w) \le R_{1*}\}; \tag{3.22}$$

$$\mathcal{S}_3 := \mathcal{S}_3^+ \cup \mathcal{S}_3^-, \tag{3.23}$$

where

$$\mathcal{S}_3^+ := \{(u,w) \in \mathcal{S} \cap A^+ | R_1^* < \mathcal{B}(u,w)\}, \tag{3.24}$$

$$\begin{aligned} \mathcal{S}_3^- : &= \{(u,w) \in \mathcal{S} \cap A^+ | R_{1*} < \mathcal{B}(u,w) < r_1^*\} \\ &\cup \{(u,w) \in \mathcal{S} \cap A^- | R_{1*} < \mathcal{B}(u,w)\}; \end{aligned} \tag{3.25}$$

In the case of (4) we define

$$\mathcal{S}_1 := \mathcal{S}_0 \cup \mathcal{S}_1^+ \cup \mathcal{S}_1^-,$$

where

$$\mathcal{S}_1^+ := \{(u,w) \in \mathcal{S} \cap A^+ | r_0^* < \mathcal{B}(u,w) < r_1^*\},$$

$$\mathcal{S}_1^- := \{(u,w) \in \mathcal{S} \cap A^- | r_0^* < \mathcal{B}(u,w) < r_{1*}\};$$

$$\mathcal{S}_2 := \mathcal{S}_2^+ \cup \mathcal{S}_2^-,$$

where

$$\mathcal{S}_2^+ := \{(u,w) \in \mathcal{S} \cap A^+ | r_1^* \le \mathcal{B}(u,w) \le R_1^*\},$$

$$\mathcal{S}_2^- := \{(u,w) \in \mathcal{S} \cap A^- | r_{1*} \le \mathcal{B}(u,w) \le R_{1*}\};$$

$$\mathcal{S}_3 := \mathcal{S}_3^+ \cup \mathcal{S}_3^-,$$

where

$$\mathcal{S}_3^+ := \{(u,w) \in \mathcal{S} \cap A^+ | R_1^* < \mathcal{B}(u,w)\},$$

$$\mathcal{S}_3^- := \{(u,w) \in \mathcal{S} \cap A^- | R_{1*} < \mathcal{B}(u,w)\}.$$

In the case of (5) we define

$$\mathcal{S}_1 := \mathcal{S}_0 \cup \mathcal{S}_1^0,$$

where

$$\mathcal{S}_1^0 := \{(u,w) \in \mathcal{S} | r_0^* < \mathcal{B}(u,w) < r_1^*\};$$

$$\mathcal{S}_2 := \mathcal{S}_2^+ \cup \mathcal{S}_2^-,$$

where

$$\begin{aligned}\mathcal{S}_2^+ : &= \{(u,w) \in \mathcal{S} \cap A^+ | r_1^* \leq \mathcal{B}(u,w) \leq R_1^*\} \\ &\cup \{(u,w) \in \mathcal{S} \cap A^- | r_1^* \leq \mathcal{B}(u,w) < r_{1*}\};\end{aligned}$$

$$\mathcal{S}_2^- := \{(u,w) \in \mathcal{S} \cap A^- | r_{1*} \leq \mathcal{B}(u,w) \leq R_{1*}\};$$

$$\mathcal{S}_3 := \mathcal{S}_3^+ \cup \mathcal{S}_3^-,$$

where

$$\mathcal{S}_3^+ := \{(u,w) \in \mathcal{S} \cap A^+ | R_1^* < \mathcal{B}(u,w)\},$$

$$\mathcal{S}_3^- := \{(u,w) \in \mathcal{S} \cap A^- | R_{1*} < \mathcal{B}(u,w)\}.$$

In the case of (6) we define

$$\mathcal{S}_1 := \mathcal{S}_0 \cup \mathcal{S}_1^0,$$

where

$$\mathcal{S}_1^0 := \{(u,w) \in \mathcal{S} | r_0^* < \mathcal{B}(u,w) < r_1^*\};$$

$$\mathcal{S}_2 := \mathcal{S}_2^+ \cup \mathcal{S}_2^-,$$

where

$$\mathcal{S}_2^+ := \{(u,w) \in \mathcal{S} | r_1^* \leq \mathcal{B}(u,w) \leq R_1^*\},$$

$$\mathcal{S}_2^- := \{(u,w) \in \mathcal{S} \cap A^- | r_{1*} \leq \mathcal{B}(u,w) \leq R_{1*}\};$$

$$\mathcal{S}_3 := \mathcal{S}_3^+ \cup \mathcal{S}_3^-,$$

where

$$\begin{aligned}\mathcal{S}_3^+ : &= \{(u,w) \in \mathcal{S} \cap A^+ | R_1^* < \mathcal{B}(u,w)\} \\ &\cup \{(u,w) \in \mathcal{S} \cap A^- | R_1^* < \mathcal{B}(u,w) < r_{1*}\},\end{aligned}$$

$$\mathcal{S}_3^- := \{(u,w) \in \mathcal{S} \cap A^- | R_{1*} < \mathcal{B}(u,w)\}.$$

In any cases of (1)-(6), when the initial data belong to any subset of $\mathcal{S}_1, \mathcal{S}_2$ and $\mathcal{S}_3$, the orbits of the solutions satisfy the statements (i)-(iii) of Theorem 2.2. In the next section, we prepare some Lemmas in order to prove Theorem 2.2.

4 Local behaviour of orbits

In this section, we investigate the local behaviour of the orbit $(u(t), w(t))$, satisfying

$$aw'(t) + bu'(t) + \partial I_{u(t)}(w(t)) \ni \alpha u(t) + \beta w(t),$$
$$cw'(t) + du'(t) = \gamma u(t) + \delta w(t)$$

for $t \geq 0$. We only give proof of Lemma 4.3. Other Lemmas are shown without proofs.

Lemma 4.1 *Assume that $(u(t_1), w(t_1))$, $t_1 \geq 0$, is in the interior of $\mathcal{S}$. Then:*

(a) if $\mathcal{B}(u(t_1), w(t_1)) \leq r_0$, then $\{u, w\}$ satisfies

$$\begin{aligned} u'(t) &= -\frac{c\beta - a\delta}{ad - bc} w, \\ w'(t) &= \frac{d\alpha - b\gamma}{ad - bc} u. \end{aligned} \tag{4.1}$$

for all $t \geq t_1$, and hence the orbit $(u(t), w(t))$ draws the anticlockwise ellipse $C_1 := \{(u, w) | \mathcal{B}(u, w) = \mathcal{B}(u(t_1), w(t_1))\}$ and it is periodic in time on $[t_1, +\infty)$.

(b) if $\mathcal{B}(u(t_1), w(t_1)) > r_0$, then $\{u, w\}$ satisfies system (4.1) on a compact interval $[t_1, t_2]$ with $t_2 > t_1$, where t_2 is the earliest time of all t $(> t_1)$ at which $(u(t), w(t)) \in \Gamma_* \cup \Gamma^*$. Hence the orbit $(u(t), w(t))$ draws an anticlockwise arc on the ellipse C_1 for $t_1 \leq t \leq t_2$.

We note that the stationary problem of (1.1)-(1.2) is of the form

$$\partial I_u(w) \ni \alpha u + \beta w, \ \ \gamma u + \delta w = 0.$$

Lemma 4.2 *(a) Let $(\tilde{u}, \tilde{w})$ be an interior point of $\mathcal{S}$. Then $\{\tilde{u}, \tilde{w}\}$ is a stationary solution of (1.1)-(1.2) if and only if $\tilde{u} = 0$ and $\tilde{w} = 0$.*

(b) Let $(\tilde{u}, \tilde{w})$ be a boundary point of $\mathcal{S}$. Then $\{\tilde{u}, \tilde{w}\}$ is a stationary solution of (1.1)-(1.2) if and only if $(\tilde{u}, \tilde{w}) \in \Gamma_*(l) \cup \Gamma^*(l)$.

Lemma 4.3 *Assume that $(u(t_1), w(t_1))$, $t_1 \geq 0$, is on Γ_* and $w(t_1) < 0$. Then:*

(a) if $\gamma u(t_1) + \delta w(t_1) > 0$ and if there exists $\bar{u} > u(t_1)$ such that

$$\gamma v + \delta f_*(v) > 0 \text{ for } u(t_1) \leq v \leq \bar{u},$$

and moreover if

$$\frac{(d\alpha - b\gamma)u}{(a\delta - c\beta)f_*(u)} \leq f_*'(u) \text{ for } u(t_1) \leq v \leq \bar{u}, \tag{4.2}$$

then $\{u, w\}$ *satisfies*

$$u'(t) = \frac{\gamma u(t) + \delta f_*(u(t))}{c f_*'(u(t)) + d}, \quad w'(t) = f_*'(u(t))u'(t) \tag{4.3}$$

on a compact interval $[t_1, t_2]$, *where* t_2 *is the earliest time at which* $u(t_2) = \bar{u}$, *and the orbit* $(u(t), w(t))$ *moves along* Γ_* *from* $(u(t_1), w(t_1))$ *to* $(\bar{u}, f_*(\bar{u}))$ *for* $t_1 \le t \le t_2$. *Moreover*

$$\frac{d}{dt}\mathcal{B}(u(t), w(t)) \le 0 \text{ on } [t_1, t_2]. \tag{4.4}$$

(b) *if* $\gamma u(t_1) + \delta w(t_1) > 0$ *and if there exists a stationary point* $(\bar{u}, \bar{w}) \in \Gamma_*(l)$ *with* $\bar{u} > u(t_1)$ *such that*

$$\gamma v + \delta f_*(v) > 0 \text{ for } u(t_1) \le v < \bar{u},$$

then $\{u, w\}$ *satisfies (4.3) on* $[t_1, +\infty)$, *and the orbit* $(u(t), w(t))$ *moves upward along the curve* Γ_* *and converges to* $(\bar{u}, \bar{w})$ *as* $t \to +\infty$;

(c) *if* $\gamma u(t_1) + \delta w(t_1) < 0$ *and if there exists a stationary point* $(\underline{u}, \underline{w}) \in \Gamma_*(l)$ *with* $\underline{u} < u(t_1)$ *such that*

$$\gamma v + \delta f_*(v) < 0 \text{ for } \underline{u} < v \le u(t_1),$$

then $\{u, w\}$ *satisfies (4.3) on* $[t_1, +\infty)$, *and the orbit* $(u(t), w(t))$ *moves downward along the curve* Γ_* *and converges to* $(\underline{u}, \underline{w})$ *as* $t \to +\infty$.

(d) *if* $\gamma v + \delta f_*(v) < 0$ *holds for all* $v \le u(t_1)$, *then* $\{u, w\}$ *satisfies (4.3) on* $[t_1, +\infty)$, *and the orbit* $(u(t), w(t))$ *diverges to* $(-\infty, f_\infty)$ *as* $t \to +\infty$.

Proof. We prove (a). We put $(u_1, w_1) = (u(t_1), w(t_1))$; note that $w_1 = f_*(u_1)$, since $(u_1, w_1) \in \Gamma_*$. We can find a positive constant M such that

$$\gamma v + \delta f_*(v) \ge M \text{ for } u_1 \le v \le \bar{u}. \tag{4.5}$$

Now, consider the Cauchy problem

$$\hat{u}'(t) = \frac{\gamma \hat{u}(t) + \delta f_*(\hat{u}(t))}{c f_*'(\hat{u}(t)) + d}, \quad t_1 \le t < t_1^*, \tag{4.6}$$

$$\hat{u}(t_1) = u_1 \tag{4.7}$$

where t_1^* is the supremum of positive number $t_1'(> t_1)$ such that problem (4.6)-(4.7) has a solution on $[t_1, t_1']$. In fact, since the function $v \longmapsto \frac{\gamma v + \delta f_*(v)}{c f_*'(v) + d}$ is Lipschitz continuous in a neighborhood of $v = u_1$, by the general theory of ODEs the problem (4.6)-(4.7) has a (unique) local (in time) solution $\hat{u}(t)$. It is easy to see from (4.5) that $\hat{u}(\cdot)$ is monotonically increasing and reaches the value $\bar{u}$ in a finite time $t_2 \in (t_1, t_1^*)$. Now, putting $\hat{w}(t) = f_*(\hat{u}(t))$ on $[t_1, t_2]$, we have that $\{\hat{u}, \hat{w}\}$ satisfies our system (1.1) and (1.2) on $[t_1, t_2]$. In fact, it follows from (4.6) that

$$cf_*'(\hat{u}(t))\hat{u}'(t) + d\hat{u}'(t) = \gamma\hat{u}(t) + \delta f_*(\hat{u}(t)),$$

which implies $c\hat{w}'(t)+d\hat{u}'(t) = \gamma\hat{u}(t)+\delta\hat{w}(t)$ on $[t_1, t_2]$. Thus (1.2) is satisfied. Equation (1.1) is checked as follows. By assumption (A1) and (4.2), calculating $\alpha\hat{u}+\beta\hat{w}-a\hat{w}'-b\hat{u}'$, we obtain

$$\begin{aligned}
\alpha\hat{u} + \beta\hat{w} - a\hat{w}' - b\hat{u}' &= \alpha\hat{u} + \beta f_*(\hat{u}) - \frac{\gamma\hat{u} + \delta f_*(\hat{u})}{cf_*'(\hat{u}) + d}(af_*'(\hat{u}) + b) \\
&= \frac{(\alpha\hat{u} + \beta f_*(\hat{u}))(cf_*'(\hat{u}) + d) - (\gamma\hat{u} + \delta f_*(\hat{u}))(af_*'(\hat{u}) + b)}{cf_*'(\hat{u}) + d} \\
&= \frac{\{(c\alpha - a\gamma)\hat{u} + (c\beta - a\delta)f_*(\hat{u})\}f_*'(\hat{u}) - (b\gamma - d\alpha)\hat{u} - (b\delta - d\beta)f_*(\hat{u})}{cf_*'(\hat{u}) + d} \\
&= \frac{(c\beta - a\delta)f_*(u)f_*'(u) - (b\gamma - d\alpha)u}{cf_*'(u) + d} \\
&\leq 0
\end{aligned}$$

on $[t_1, t_2]$. By the definition of subdifferentials (see (1.4)) we have $\partial I_{\hat{u}}(\hat{w}) = (-\infty, 0]$ for $\hat{w} = f_*(\hat{u})$. Therefore

$$\alpha\hat{u} + \beta\hat{w} - a\hat{w}' - b\hat{u}' \in \partial I_{\hat{u}}(\hat{w}) \text{ on } [t_1, t_2].$$

Thus, by the uniqueness, $\{\hat{u}, \hat{w}\}$ must be the solution $\{u, w\}$ of (1.1)-(1.2) on $[t_1, t_2]$. Next we show (4.4). Since (4.2) and (4.3) hold on $[t_1, t_2]$, we obtain

$$\begin{aligned}
\frac{d}{dt}\mathcal{B}(u, w) &= \frac{u'}{\mathcal{B}(u, w)}\{(c\beta - a\delta)f_*'(u)f_*(u) - (b\gamma - d\alpha)u\} \\
&\leq 0 \text{ on } [t_1, t_2].
\end{aligned}$$

Next we prove (b). Let us recall that $\bar{u} < 0, f_*(\bar{u}) < 0$ by Lemma 4.2 (b). We obtain automatically

$$\frac{(d\alpha - b\gamma)v}{(a\delta - c\beta)f_*(v)} \leq f_*'(v) \text{ for } u_1 \leq v \leq \bar{u}. \tag{4.8}$$

Therefore, in the same way as in (a), $\{u, w\}$ satisfies (4.3) for a moment after the time t_1 and the orbit $(u(t), w(t))$ moves along the curve Γ_* starting from $(u(t_1), w(t_1))$. We now show that $(u(t), w(t))$ converges to $(\bar{u}, \bar{w}) \in \Gamma_*(l)$ as $t \to +\infty$. Let T be the supremum of all $s(\geq t_1)$ such that

$$u'(t) = \frac{\gamma u(t) + \delta f_*(u(t))}{c f_*'(u(t)) + d}, \ w(t) = f_*(u(t)) \text{ for } \forall t \in [t_1, s].$$

Then, just as in the case of (a), we see that $T > t_1$. Since u is non-decreasing on $[t_1, T)$, $\lim_{t \nearrow T} u(t)$ exists. We want to see that $\lim_{t \nearrow T} u(t) = \bar{u}$. We show it by contradiction. Now, assume that $\lim_{t \nearrow T} u(t) < \bar{u}$. Then we consider the following statements:

(i) $T = +\infty$, $u_\infty := \lim_{t \to +\infty} u(t)$ and $w_\infty := \lim_{t \to +\infty} w(t)$ give a pair of stationary solutions

or

(ii) $T < \infty$ and $\frac{(d\alpha - b\gamma)u(t)}{(a\delta - c\beta) f_*(u(t))} > f_*'(u(t))$ for some $t > T$.

But these cases do not occur in our situations considered now. In fact, the case (i) yields that $u(t_1) \leq u_\infty < \bar{u}$ and $\gamma u_\infty + \delta f_*(u_\infty) = 0$, which contradicts our assumption. Also, the case (ii) yields a contradiction to (4.8).

Assertion (c) is similarly proved to (b).

Finaly we prove (d). By the same argument as above, we have

$$\frac{(d\alpha - b\gamma)v}{(a\delta - c\beta) f_*(v)} \leq f_*'(v) \text{ for all } v \leq u_1,$$

and find a negative constant $\tilde{M}$ such that

$$\gamma v + \delta f_*(v) \leq \tilde{M} \text{ for all } v \leq u_1.$$

Hence $\{u, w\}$ satisfies (4.3) for all $t \geq t_1$ and $u(\cdot)$ is monotonically decreasing on $[t_1, \infty)$. By assumption (A2), $(u(t), w(t))$ diverges to $(-\infty, f_\infty)$ as $t \to +\infty$. ∎

Lemma 4.4 *Assume that $(u(t_1), w(t_1))$, $t_1 \geq 0$, is on Γ^* and $w(t_1) > 0$. Then:*

(a) if $\gamma u(t_1) + \delta w(t_1) < 0$ and if there exists $\bar{u} < u(t_1)$ such that

$$\gamma v + \delta f^*(v) < 0 \textit{ for } \bar{u} \leq u(t_1) \leq v,$$

and moreover if the following condition hold that

$$\frac{(d\alpha - b\gamma)v}{(a\delta - c\beta) f^*(v)} \leq f^{*\prime}(v) \textit{ for } \bar{u} \leq v \leq u(t_1),$$

then $\{u,w\}$ *satisfies*

$$u'(t)=\frac{\gamma u+\delta f^*(u)}{cf^{*\prime}(u)+d},\ w'(t)=f^{*\prime}(u)u'(t)$$

on a compact interval $[t_1,t_2]$*, where* t_2 *is the earliest time at which* $u(t_2)=\bar{u}$*, and the orbit* $(u(t),w(t))$ *moves along* Γ_* *from* $(u(t_1),w(t_1))$ *to* $(\bar{u},f^*(\bar{u}))$ *for* $t_1\le t\le t_2$*. Moreover*

$$\frac{d}{dt}\mathcal{B}(u,w)\le 0 \ on\ [t_1,t_2].$$

(b) if $\gamma u(t_1)+\delta w(t_1)<0$ *and if there exists a stationary point* $(\bar{u},\bar{w})\in\Gamma^*(l)$ *with* $\bar{u}<u(t_1)$ *such that*

$$\gamma v+\delta f_*(v)<0 \ for\ \bar{u}<v\le u(t_1),$$

then $\{u,w\}$ *satisfies (4.4) on* $[t_1,+\infty)$*, and the orbit* $(u(t),w(t))$ *moves downward along the curve* Γ^* *and converges to* $(\bar{u},\bar{w})$ *as* $t\to+\infty$*.*

(c) if $\gamma u(t_1)+\delta w(t_1)>0$ *and if there exists a stationary point* $(\underline{u},\underline{w})\in\Gamma^*(l)$ *with* $\underline{u}>u(t_1)$ *such that*

$$\gamma v+\delta f_*(v)>0 \ for\ u(t_1)\le v<\underline{u},$$

then $\{u,w\}$ *satisfies (4.4) for* $[t_1,+\infty)$*. Hence the orbit* $(u(t),w(t))$ *moves upward along the curve* Γ^* *and converges to* $(\underline{u},\underline{w})$ *as* $t\to+\infty$*.*

(d) if $\gamma v+\delta f_*(v)>0$ *holds for all* $v\ge u(t_1)$*, then* $\{u,w\}$ *satisfies (4.4) for* $[t_1,+\infty)$*. Hence the orbit* $(u(t),w(t))$ *diverges to* (∞,f^∞) *as* $t\to+\infty$*.*

5 Large time behaviour of orbits

In this section, we prove Theorem 2.2 in the case (1) in section 3. Any other cases can be treated by a simple modification of them. We investigate the behaviour of the solution $\{u,w\}$ when the initial data (u_0,w_0) belong to each of $\mathcal{S}_0,\mathcal{S}_1,\mathcal{S}_2$ and $\mathcal{S}_3$.

In the case of $(u_0,w_0)\in\mathcal{S}_0$

When $(u_0,w_0)\in\mathcal{S}_0$, we obtain $\mathcal{B}(u_0,w_0)\le r_0$. Therefore, by Lemma 4.1(a), we see that the orbit $(u(t),w(t))$ draws anticlockwise ellipse $\mathcal{B}(u,w)=\mathcal{B}(u_0,w_0)$ for all $t\ge 0$, and is periodic in time.

In the case of $(u_0,w_0)\in\mathcal{S}_1$

First, we consider the case of $(u_0,w_0)\in\mathcal{S}_1^-$ and $w_0\le 0$. Clearly $\mathcal{B}(u_0,w_0)>r_0$.

By Lemma 4.1 (b), the orbit $(u(t), w(t))$ draws an anticlockwise ellipse on $\mathcal{B}(u,w) = \mathcal{B}(u_0, w_0)$, untill it reaches Γ_*, satisfying

$$u'(t) = -\frac{c\beta - a\delta}{ad - bc} w(t), \ 0 \le t \le t_1,$$

$$w'(t) = \frac{d\alpha - b\gamma}{ad - bc} u(t), \ 0 \le t \le t_1,$$

$$u(0) = u_0, \ w(0) = w_0,$$

where t_1 is the earliest time such that $(u(t_1), w(t_1)) \in \Gamma_*$. We have $w(t_1) = f_*(u(t_1))$, $\mathcal{B}(u(t_1), w(t_1)) < r_{1*}$ and

$$\gamma v + \delta f_*(v) > 0 \text{ for } u(t_1) \le v \le u_*.$$

Next, take the number u_2 so that

$$u_2 = \sup\left\{ \tilde{u} \,\middle|\, u(t_1) \le \tilde{u} \le u_*, \ \frac{(d\alpha - b\gamma)u}{(a\delta - c\beta) f_*(u)} \le f_*'(u) \text{ for } \forall u \in [u(t_1), \tilde{u}] \right\}.$$

Then we have the following three possibilities: (i) $u(t_1) < u_2 < u_*$, (ii) $u_2 = u(t_1)$, (iii) $u_2 = u_*$.

In the case of (i), by Lemma 4.3 (a)

$$u'(t) = \frac{\gamma u(t) + \delta f_*(u(t))}{c f_*'(u(t)) + d}, \ w(t) = f_*(u(t)), \ t \in [t_1, t_2]$$

where t_2 is the earliest time such that $u(t_2) = u_2$. We denote by C_2 the ellipse $\mathcal{B}(u,w) = \mathcal{B}(u_2, f_*(u_2)) =: r_2$. By assumption (A3) and the definition of u_2, we see that an arc $\{(u,w) | u_2 \le u \le \tilde{u}_3, \ \mathcal{B}(u,w) = r_2\}$ on C_2 is contained in $\mathcal{S}$. Now, denote by u_3 the largest one of such numbers $\tilde{u}_3$, we have $u_3 > u_2$. Moreover, by Lemma 4.1 (b), $\{u, w\}$ is given by

$$u'(t) = -\frac{c\beta - a\delta}{ad - bc} w(t), \ w'(t) = \frac{d\alpha - b\gamma}{ad - bc} u(t), \ t \in [t_2, t_3],$$

where t_3 is the earliest time such that $u(t_3) = u_3$. Our assumption (A3) guarantees that the orbit $(u(t), w(t))$ reaches $(u_*, f_*(u_*))$ at $t = t_* (< \infty)$ by repeating finitely many times such behaviours as above. Here, after the time t_*, the orbit $(u(t), w(t))$ draws the anticlockwise ellipse $\mathcal{B}(u,w) = r_0$ periodically in time (see Lemma 4.1 (a)).

In the case of (ii), it is the case that $t_1 = t_2$ with the same notation as above, and the behaviour of $(u(t), w(t))$ is similar to the case of (i) after the time t_2.

In the case of (iii), it is the case that $t_2 = t_*$, and the behaviour of $(u(t), w(t))$ is the anticlockwise ellipse $\mathcal{B}(u,w)=r_0$ after the time t_*.

Next, consider the case of $(u_0,w_0)\in\mathcal{S}_1^-$ with $w_0>0$. In this case, the orbit $(u(t),w(t))$ draws an anticlockwise arc on the ellipse $\mathcal{B}(u,w)=\mathcal{B}(u_0,w_0)$ untill it reaches Γ_* or Γ^* at time s_1. If $(u(s_1),w(s_1))\in\Gamma_*$, then the behaviour of $(u(t),w(t))$ is exactly the same as in the previous case after time s_1. On the other hand, if $(u(s_1),w(s_1))\in\Gamma^*$, then the orbit $(u(t),w(t))$ moves downward for a time interval $[s_1,s_2]$ with $s_1\le s_2$ along the curve Γ^* by Lemma 4.4 (a) (in this step assumption (A3) regarding the function $f^*(\cdot)$ is used), where s_2 is the largest time of $\tilde{s}_2$ such that $(u(t),w(t))\in\Gamma^*$ for $\forall t\in[s_1,\tilde{s}_2]$. It is easy to see that $w(s_2)>0$ and $s_2<+\infty$. After time s_2, the orbit $(u(t),w(t))$ draws an anticlockwise arc on $\mathcal{B}(u,w)=\mathcal{B}(u(s_2),w(s_2))$ untill it reaches Γ_* or Γ^* at time s_3. Repeating such procedures finitely many times, the orbit $(u(t),w(t))$ arrives at Γ_* at time $t=t_1$ in the last step. After time t_1, the behaviour of $(u(t),w(t))$ was already seen in the case of $(u_0,w_0)\in\mathcal{S}_1^-$ with $w_0\le 0$.

Finaly, we consider the case of $(u_0,w_0)\in\mathcal{S}_1^+$. We have the following three cases:

(i) $(u_0,w_0)\in\mathcal{S}_1^+$ with $\mathcal{B}(u_0,w_0)\ge r_0^*$,

(ii) $(u_0,w_0)\in\mathcal{S}_1^+$ with $\mathcal{B}(u_0,w_0)<r_0^*$ and $w_0\ge 0$,

(iii) $(u_0,w_0)\in\mathcal{S}_1^+$ with $\mathcal{B}(u_0,w_0)<r_0^*$ and $w_0<0$.

First, we consider the case (i). In this case, the orbit $(u(t),w(t))$ draws an anticlockwise arc on the ellipse $\mathcal{B}(u,w)=r\in[r_0^*,r_1^*)$ and a part of Γ^* alternately and reaches the point $(u^*,f^*(u^*))$ at a finite time $t=t^*$. Since $(u^*,f^*(u^*))\in\mathcal{S}_1^-$, the behaviour of $(u(t),w(t))$ after the time t^* is the same as in the case $(u_0,w_0)\in\mathcal{S}_1^-$ with $w_0>0$.

In the second case (ii), the orbit $(u(t),w(t))$ draws an anticlockwise arc on the ellipse $\mathcal{B}(u,w)=\mathcal{B}(u_0,w_0)$ and reaches a point $(u_0,w_0)\in\Gamma_*$ with $u_1<u_*$ and $w_1<0$ at a time $t=t_1$. After the time t_1, the behaviour of $(u(t),w(t))$ is the same as in the case $(u_0,w_0)\in\mathcal{S}_1^-$ with $w_0<0$.

In the third case (iii), the orbit $(u(t),w(t))$ possibly draws an anticlockwise arc on the ellipse $\mathcal{B}(u,w)=r\in(r_0,r_0^*)$ and a part of Γ_* alternately and reaches a point $(u_1,w_1)\in\Gamma_*$ with $u_1<u_*$ and $w_1<0$ at a finite time $t=t_1$. After the time t_1, the behaviour of $(u(t),w(t))$ is the same as the case $(u_0,w_0)\in\mathcal{S}_1^-$ with $w_0<0$.

In the case of $(u_0,w_0)\in\mathcal{S}_2$

We give a proof only in the case of $(u_0,w_0)\in\mathcal{S}_2^-$, since the proof of the case of $(u_0,w_0)\in\mathcal{S}_2^+$ is quite similar. In a way similar to that in the case of $(u_0,w_0)\in\mathcal{S}_1$, we see that the orbit $(u(t),w(t))$, drawing an anticlockwise arc on the ellipse $\mathcal{B}(u,w)=r\in[r_{1*},R_{1*}]$, arrives at a point $(u_1,w_1)\in\Gamma_*$ at a certain finite time $t=t_1$. If $(u(t_1),w(t_1))(=(u_1,w_1))\in\Gamma_*(l)$, then $(u(t_1),w(t_1))$ is a stationary solution of (1.1)-(1.3) by Lemma 4.2 (b). If $(u(t_1),w(t_1))\notin\Gamma_*(l)$, then we have the following two cases:

(i) $\gamma u(t_1) + \delta w(t_1) > 0$,

(ii) $\gamma u(t_1) + \delta w(t_1) < 0$.

Suppose now that (i) holds. Then there is a closed interval $[\underline{u}, \bar{u}] \subset (-\infty, 0)$ on the u-axis such that $\underline{u} < u(t_1) < \bar{u}$, $\gamma v + \delta f_*(v) > 0$ for all $v \in (\underline{u}, \bar{u})$ and $\gamma \underline{u} + \delta f_*(\underline{u}) = \gamma \bar{u} + \delta f_*(\bar{u}) = 0$. Therefore, the orbit $(u(t), w(t))$ converges to $(\bar{u}, f_*(\bar{u})) \in \Gamma_*(l)$ as $t \to +\infty$ by Lemma 4.3 (b). On the other hand, when (ii) holds, the orbit $(u(t), w(t))$ converges to a stationary point as $t \to +\infty$, too.

In the case of $(u_0, w_0) \in \mathcal{S}_3$

It is enough to consider only the case $(u_0, w_0) \in \mathcal{S}_3^-$. In the same way as in the case of $(u_0, w_0) \in \mathcal{S}_2$, the orbit $(u(t), w(t))$ reaches Γ_* in a finite time t_1. Also, we obtain $\mathcal{B}(u(t_1), w(t_1)) < R_{1*}$ and $\gamma v + \delta f_*(v) < 0$ for $v < u_1$. Therefore, by Lemma 4.3 (d), we see that $(u(t), w(t))$ diverges to $(-\infty, f_\infty)$ as $t \to +\infty$. Similarly, in the case $(u_0, w_0) \in \mathcal{S}_3^+$, we see that $(u(t), w(t))$ diverges to (∞, f^∞) as $t \to +\infty$.

Remark 5.1 *We have many cases about the stability around stationary points in $\mathcal{S}_2$. If, for instance, we restrict our geometry of the curves Γ_*, Γ^* and l to the one as illustrated by the picture (Fig. 1), then stationary points are classified into the following three categories: Let (u_∞, w_∞) be any stationary point in $\mathcal{S}_2$. Then one of the following cases happens.*

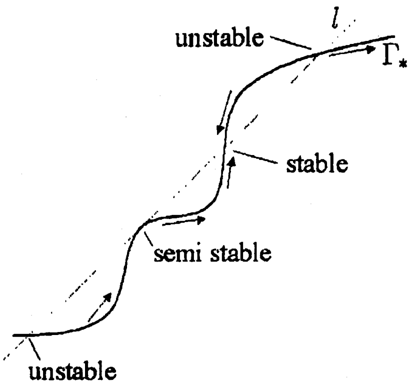

Fig. 1

(1) (u_∞, w_∞) is stable. Namely, there is a neighborhood U_1 of (u_∞, w_∞) in R^2 such that the orbit $(\tilde{u}(t), \tilde{w}(t))$ stays in $U_1 \cap \mathcal{S}$ for all $t \geq 0$ and converges to (u_∞, w_∞) as $t \to +\infty$, whenever $(\tilde{u}_0, \tilde{w}_0) (= (\tilde{u}(0), \tilde{w}(0))) \in U_1 \cap \mathcal{S}$.

(2) (u_∞, w_∞) is semistable. Namely, there is a neighborhood U_2 of (u_∞, w_∞) in R^2 such that the following properties (i) and (ii) are satisfied:

(i) For any initial point $(\hat{u}_0, \hat{w}_0) \in U_2 \cap \mathcal{S} \cap \mathcal{K}_\infty$, the orbit $(\hat{u}(t), \hat{w}(t))$ stays in $U_2 \cap \mathcal{S}$ for all $t \geq 0$ and converges to (u_∞, w_∞) as $t \to +\infty$, whenever $(\hat{u}_0, \hat{w}_0) (= (\hat{u}(0), \hat{w}(0))) \in U_2 \cap \mathcal{S}$.

(ii) For any initial point $(\bar{u}_0, \bar{w}_0) \in U_2 \cap \mathcal{S} \cap \mathcal{K}_\infty^c$, the orbit $(\bar{u}(t), \bar{w}(t))$ gets out of U_2 after a certain time t_1.

where $\mathcal{K}_\infty := \{(u,w)|\mathcal{B}(u,w) \geq \mathcal{B}(u_\infty, w_\infty)\}$.

(3) (u_∞, w_∞) *is unstable. Namely, there is a neighborhood* U_3 *of* (u_∞, w_∞) *in* R^2 *such that the following properties (iii) and (iv) are satisfies:*

(iii) For any initial point $(\hat{u}_0, \hat{w}_0) \in U_3 \cap \mathcal{S} \cap \mathcal{C}_\infty$, *the orbit* $(\hat{u}, \hat{w})$ *stays in* $U_3 \cap \mathcal{S}$ *for all* $t \geq 0$ *and converges to* (u_∞, w_∞) *in a finite time* t_1.

(iv) For any initial point $(\check{u}_0, \check{w}_0) \in U_3 \cap \mathcal{S} \cap \mathcal{C}^c_\infty$, *the orbit* $(\check{u}(t), \check{w}(t))$ *gets out of* U_3 *after a certain time* t_1.

where $\mathcal{C}_\infty := \{(u,w)|\mathcal{B}(u,w) = \mathcal{B}(u_\infty, w_\infty)\}$.

6 Some numerical simulations

In this section, we give some numerical experiments to verify Theorem 2.2. In order to catch the behaviour of solutions, we simply take the coefficients a, b, c, d and functions F, h satisfying (A1) with $d\alpha - b\gamma = c\beta - a\delta > 0$ such that the orbits of solutions are anticlockwise circles without subdifferential term $\partial I_u(w)$. Now we fix the coefficients a, b, c, d and functions F, h as follows:

$$a = 1,\ b = -1,\ c = 1,\ d = 1,\ F(u,w) = u + w,\ h(u,w) = u - w.$$

In this case, our system is of the following form:

$$w' - u' + \partial I_u(w) \ni u + w,\ 0 < t < T,$$

$$w' + u' = u - w,\ 0 < t < T,$$

$$u(0) = u_0,\ w(0) = w_0.$$

Now let λ and Δt be small positive numbers, and n be a large natural number. Then the difference scheme for our numerical simulation is of the form

$$\begin{aligned}
&\frac{w^{k+1} - w^k}{\Delta t} - \frac{u^{k+1} - u^k}{\Delta t} + \partial I^\lambda_{u^k}(w^{k+1}) = u^k + w^k,\\
&\frac{w^{k+1} - w^k}{\Delta t} + \frac{u^{k+1} - u^k}{\Delta t} = u^k - w^k,\ k = 0, 1, 2, \cdots,\\
&u^0 = u_0,\ w^0 = w_0,
\end{aligned}$$

where

$$\partial I^\lambda_{u^k}(w^{k+1}) = \frac{[w^{k+1} - f^*(u^k)]^+}{\lambda} - \frac{[f_*(u^k) - w^{k+1}]^+}{\lambda}.$$

The graphs of I^λ_u and ∂I^λ_u are illustrated in Figures 2 and 3, respectively.

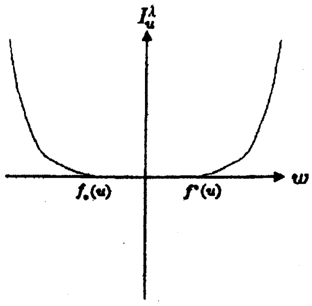

Fig. 2

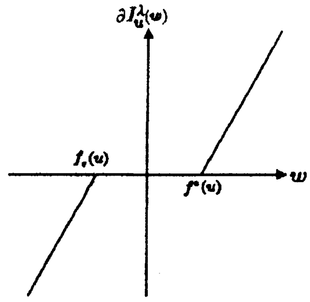

Fig. 3

In our actual computation

$$\Delta t = \frac{1}{1000}, \ \lambda = \frac{1}{1000},$$

and we examine the following items:

- We define the subset $\mathcal{S}_i (i = 0, 1, 2, 3)$ by the geometries of the given functions $f_*(u)$ and $f^*(u)$ and the line $\gamma u + \delta w = 0$.
- By numerical simulations, we verify that the behaviour of solutions satisfies the statements of Theorem 2.2 when the initial data belong to each subset $\mathcal{S}_i (i = 0, 1, 2, 3)$.

Experiment 1:
We take the functions $f_*(u), f^*(u)$ as follows:

$$f_*(u) = \begin{cases} -1 & \text{if } u \leq 0.4, \\ 5u^2 - 4u - 0.2 & \text{if } 0.4 < u \leq 0.6, \\ 2u - 2 & \text{if } 0.6 < u \leq 1.4, \\ -5u^2 + 16u - 11.8 & \text{if } 1.4 < u \leq 1.6, \\ 1 & \text{if } 1.6 < u, \end{cases} \quad f^*(u) = \begin{cases} -1 & \text{if } u \leq -1.6, \\ 5u^2 + 16u + 11.8 & \text{if } -1.6 < u \leq -1.4, \\ 2u + 2 & \text{if } -1.4 < u \leq -0.6, \\ -5u^2 - 4u + 0.2 & \text{if } -0.6 < u \leq -0.4, \\ 1 & \text{if } -0.4 < u. \end{cases}$$

$f_*(u)$ and $f^*(u)$ are symmetric with respect to origin. In this case, by our choice of $f_*(u), f^*(u)$ and the line $u + w = 0$, we obtain that

$$r_0 := r_{0*} = r_0^* = \frac{2\sqrt{5}}{5}, \ r_{1*} = r_1^* = R_{1*} = R_1^* = \sqrt{2}$$

and

stationary points are $(1, 1)$, $(0, 0)$ and $(-1, -1)$.

Therefore, subsets $\mathcal{S}_i (i = 0, 1, 2, 3)$ are defined by (3.1)-(3.9) and are illustrated by Figure. 4. Now we take the initial data which belong to each subset $\mathcal{S}_i$ and numerical experiments are shown as follows:

data	u_0	w_0	subset
Fig. 5	-0.2	-0.6	$(u_0, w_0) \in \mathcal{S}_0$
Fig. 6	-0.7	-0.8	$(u_0, w_0) \in \mathcal{S}_1$

data	u_0	w_0	subset
Fig. 7	-1.3	-0.8	$(u_0, w_0) \in \mathcal{S}_3^-$
Fig. 7	1.3	0.8	$(u_0, w_0) \in \mathcal{S}_3^+$

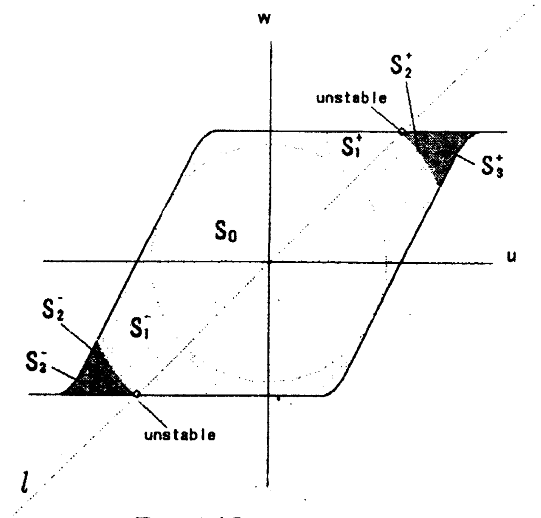

Fig. 4 ($\mathcal{S}_i$ $i = 0, 1, 2, 3$)

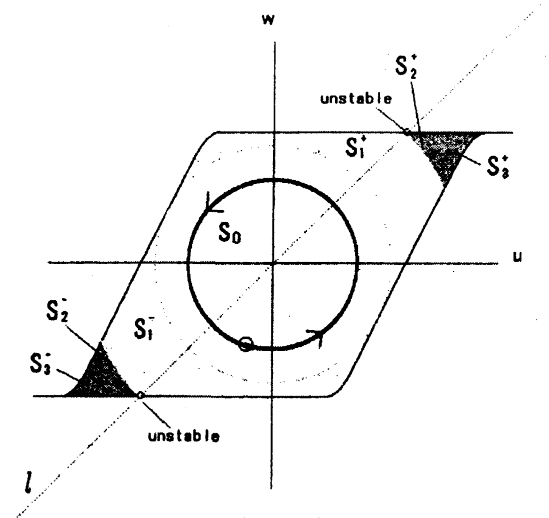

Fig. 5 $(u_0, w_0) \in \mathcal{S}_0$

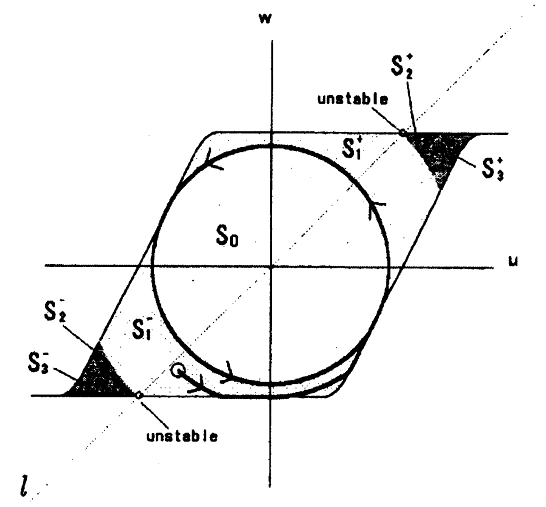

Fig. 6 $(u_0, w_0) \in \mathcal{S}_1$

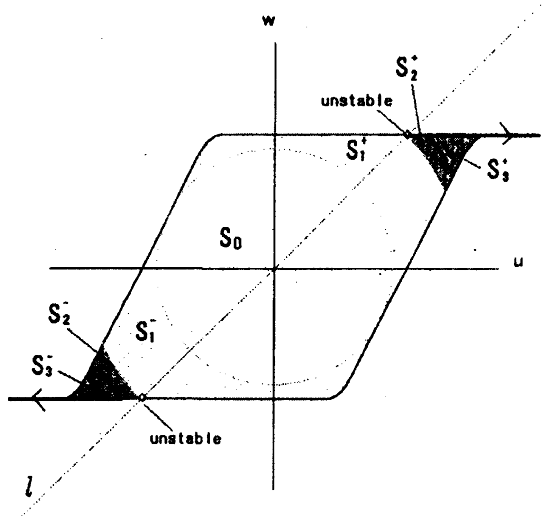

Fig. 7 $(u_0, w_0) \in \mathcal{S}_3$

When the initial data belong to $\mathcal{S}_0$, the orbit draws anticlockwise circle from the initial point (u_0, w_0) (Fig. 5). In the case when $(u_0, w_0) \in \mathcal{S}_1$, the orbit draws an anticlockwise arc and a part of Γ_* alternately and reaches a periodic circle $\mathcal{B}(u, w) = r_0$ in a finite time (Fig. 6). On the other hand, in the case when $(u_0, w_0) \in \mathcal{S}_3^-$ or $\mathcal{S}_3^+$, the orbit diverges to $(-\infty, -1)$ or $(+\infty, 1)$ as $t \to +\infty$ (Fig. 7).

Experiment 2:
We take the functions $f_*(u), f^*(u)$ as follows

$$f_*(u) = \begin{cases} -1 & \text{if } u \leq -1, \\ 3u^2 + 6u + 2 & \text{if } -1 < u \leq -0.75, \\ -u^2 - 0.25 & \text{if } -0.75 < u \leq 0, \\ -0.25 & \text{if } 0 < u \leq 0.75, \\ 4u^2 - 6u + 2 & \text{if } 0.75 < u \leq 1, \\ 2u - 2 & \text{if } 1 < u \leq 1.4, \\ -5u^2 + 16u - 11.8 & \text{if } 1.4 < u \leq 1.6, \\ 1 & \text{if } 1.6 < u. \end{cases} \qquad f^*(u) = \begin{cases} -1 & u \leq -2, \\ u^2 + 4u + 3 & \text{if } -2 < u \leq -1.5, \\ u + 0.75 & \text{if } -1.5 < u \leq 0, \\ -u^2 + u + 0.75 & \text{if } 0 < u \leq 0.5, \\ 1 & \text{if } 0.5 < u. \end{cases}$$

In this case, we obtain that

$$r_0 := r_{0*} = \frac{1}{4}, r_0^* = \frac{3\sqrt{2}}{8}, r_{1*} = \frac{\sqrt{2}}{2}, R_{1*} = \sqrt{2}, R_1^* = \sqrt{2}$$

and

stationary solutions are $(1,1)$, $(0,0)$, $(-0.5,-0.5)$ and $(-1,-1)$.

Since $r_{0*} < r_0^* < r_{1*} < R_{1*}$, $\mathcal{S}_i (i = 0,1,2,3)$ are defined by (3.1)-(3.9) and are illustrated by Figure. 8. The initial data and the subsets $\mathcal{S}_i$ in which the initial data are given in this experiments are as follows

data	u_0	w_0	subset
Fig. 9	-1.1	0.4	$(u_0, w_0) \in \mathcal{S}_1$
Fig. 10	-1.4	-0.4	$(u_0, w_0) \in \mathcal{S}_2^-$

data	u_0	w_0	subset
Fig. 11	-1.4	-0.8	$(u_0, w_0) \in \mathcal{S}_3^-$
Fig. 11	1.4	0.8	$(u_0, w_0) \in \mathcal{S}_3^+$

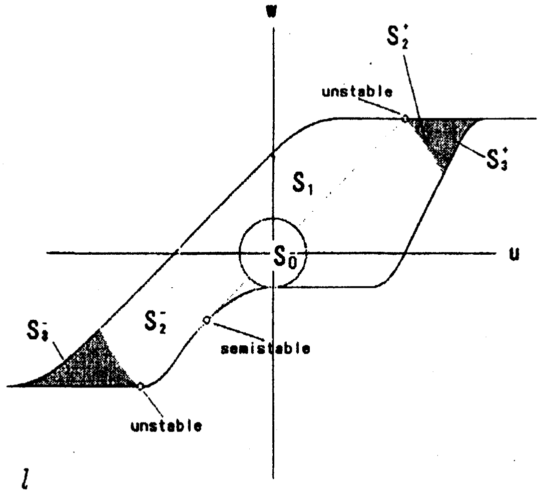

Fig. 8 ($\mathcal{S}_i$ $i = 0,1,2,3$)

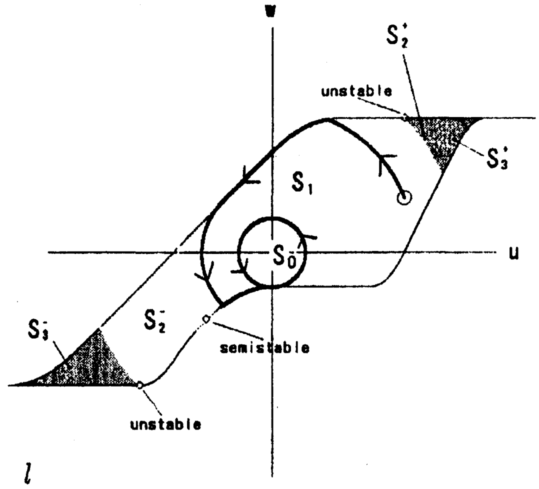

Fig. 9 $(u_0, w_0) \in \mathcal{S}_1$

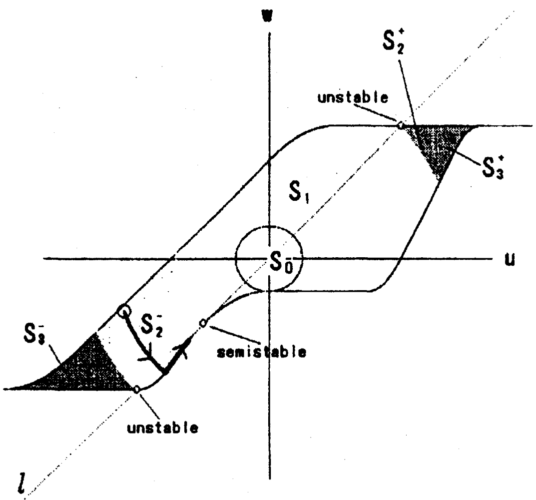

Fig. 10 $(u_0, w_0) \in \mathcal{S}_2^-$

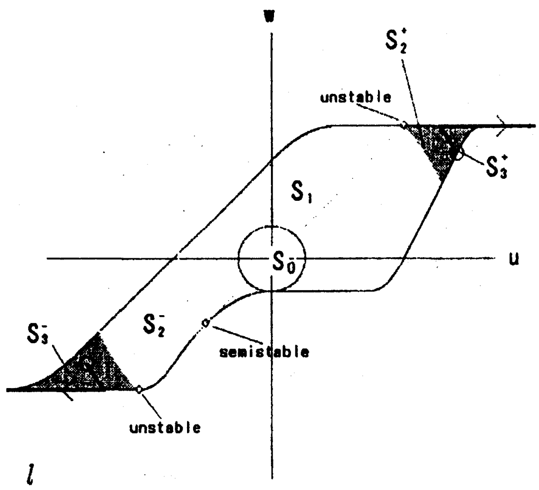

Fig. 11 $(u_0, w_0) \in \mathcal{S}_3$

We also see that the behaviour of orbits of solutions for each initial data $(u_0, w_0) \in \mathcal{S}_i (i = 0,1,2,3)$ guarantee Theorem 2.2 (Fig. 9-11). Especially by Fig. 10 and 11, we can recognize that the point $(-0.5, -0.5)$ is a semi stable stationary solution.

Experiment 3:
We take the functions $f_*(u), f^*(u)$ as follows

$$f_*(u) = \begin{cases} -1 & \text{if } u \le -1, \\ 3u^2+6u+2 & \text{if } -1 < u \le -0.75, \\ -u^2-0.25 & \text{if } -0.75 < u \le 0, \\ -0.25 & \text{if } 0 < u \le 0.75, \\ 4u^2-6u+2 & \text{if } 0.75 < u \le 1, \\ 2u-2 & \text{if } 1 < u \le 1.4, \\ -5u^2+16u-11.8 & \text{if } 1.4 < u \le 1.6, \\ 1 & \text{if } 1.6 < u. \end{cases} \qquad f^*(u) = \begin{cases} -1 & \text{if } u \le -1.6, \\ 5u^2+16u+11.8 & \text{if } -1.6 < u \le -1.4, \\ 2u+2 & \text{if } -1.4 < u \le -0.6, \\ -5u^2-4u+0.2 & \text{if } -0.6 < u \le -0.4, \\ 1 & \text{if } -0.4 < u. \end{cases}$$

In this case, we obtain that

$$r_0 := r_{0*} = \frac{1}{4}, \; r_0^* = \frac{2\sqrt{5}}{5}, \; r_{1*} = \frac{\sqrt{2}}{2}, \; R_{1*} = \sqrt{2},$$

and

stationary solutions are $(1,1)$, $(0,0)$, $(-0.5,-0.5)$ and $(-1,-1)$.

This implies that $r_{0*} < r_{1*} < r_0^* < R_{1*}$. Therefore, $\mathcal{S}_i (i = 0,1,2,3)$ are defined by (3.10)-(3.17) (Fig. 12). Given initial data, our experiments are the following (Fig. 13-15):

data	u_0	w_0	subset
Fig. 13	0.25	0.25	$(u_0, w_0) \in \mathcal{S}_1$
Fig. 14	0.8	0.8	$(u_0, w_0) \in \mathcal{S}_2^-$

data	u_0	w_0	subset
Fig. 15	−1.3	−0.8	$(u_0, w_0) \in \mathcal{S}_3^-$
Fig. 15	1.3	0.8	$(u_0, w_0) \in \mathcal{S}_3^+$

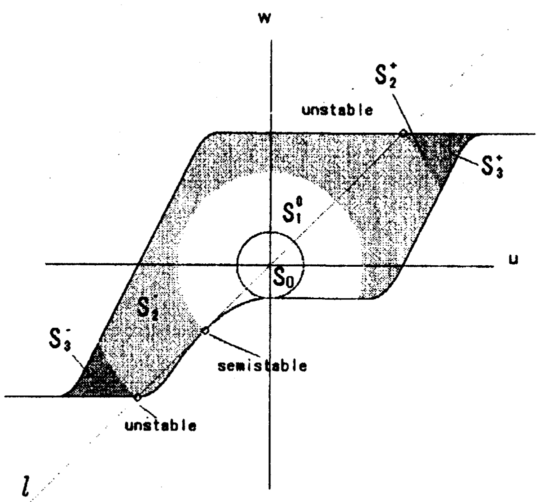

Fig. 12 ($\mathcal{S}_i\ i = 0,1,2,3$)

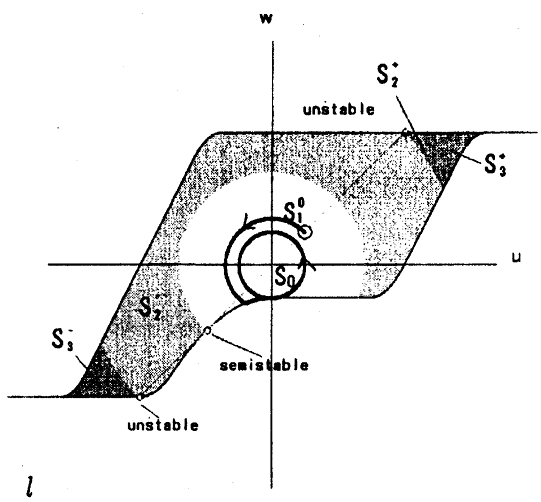

Fig. 13 $(u_0, w_0) \in \mathcal{S}_1$

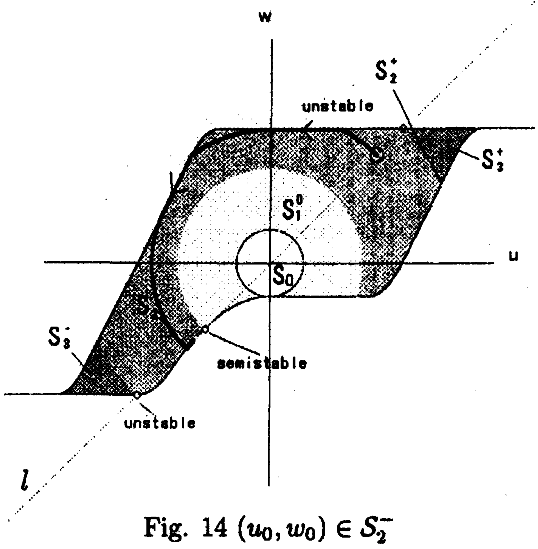

Fig. 14 $(u_0, w_0) \in S_2^-$

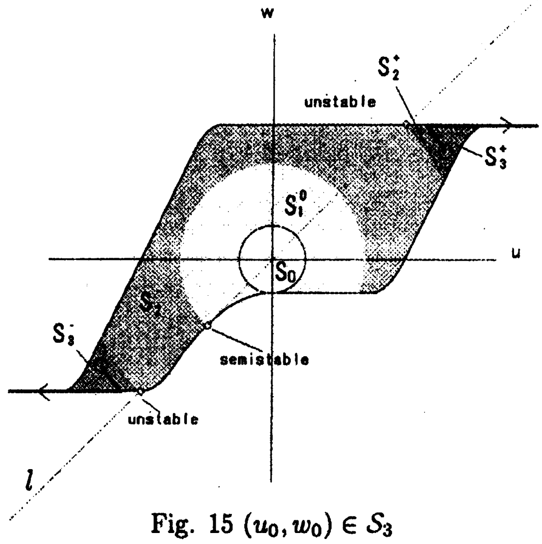

Fig. 15 $(u_0, w_0) \in S_3$

Note that the function $f_*(u)$ is the same as in experiment 2 but $f^*(u)$ is not. We see that the orbit starting from $(0.8, 0.8)$ draws an anticlockwise arc and a part of Γ^* alternately and reaches Γ_* in a finite time, and then it goes to the semi stable stationary point $(-0.5, -0.5)$ as $t \to +\infty$.

Experiment 4:
We take the functions $f_*(u), f^*(u)$ as follows

$$f_*(u) = \begin{cases} -1 & \text{if } u \le -1.25, \\ 3u^2 + 7u + 3.6875 & \text{if } -1.25 < u \le -1, \\ -u^2 - u - 0.3125 & \text{if } -1 < u \le -0.25, \\ -0.25 & \text{if } -0.25 < u \le 0.75, \\ 4u^2 - 6u + 2 & \text{if } 0.75 < u \le 1, \\ 2u - 2 & \text{if } 1 < u \le 1.4, \\ -5u^2 + 16u - 11.8 & \text{if } 1.4 < u \le 1.6, \\ 1 & \text{if } 1.6 < u. \end{cases} \qquad f^*(u) = \begin{cases} -1 & \text{if } u \le -1.6, \\ 5u^2 + 16u + 11.8 & \text{if } -1.6 < u \le -1.4, \\ 2u + 2 & \text{if } -1.4 < u \le -0.6, \\ -5u^2 - 4u + 0.2 & \text{if } -0.6 < u \le -0.4, \\ 1 & \text{if } -0.4 < u. \end{cases}$$

$f^*(u)$ is the same function as in experiment 3 and $f_*(u)$ slightly changes from the one in experiment 3. In this case, we obtain that

$$r_0 := r_{0*} = \frac{1}{16}, \; r_0^* = \frac{4}{5}, \; r_{1*} = R_{1*} = \frac{1}{8},$$

and

stationary solutions are $(1, 1)$, $(0, 0)$ and $(-0.5, -0.5)$.

Since $r_{0*} < r_{1*} = R_{1*} < r_0^*$, subset $S_i (i = 0, 1, 2, 3)$ are defined by (3.18)-(3.24) (see Fig. 16). We take the initial data as follows:

data	u_0	w_0	subset
Fig. 17	0.22	0.22	$(u_0, w_0) \in S_1$
Fig. 18	0.8	0.8	$(u_0, w_0) \in S_3^-$

data	u_0	w_0	subset
Fig. 19	1.3	0.8	$(u_0, w_0) \in S_3^+$

Then our experiments results are shown by Fig. 17-19.

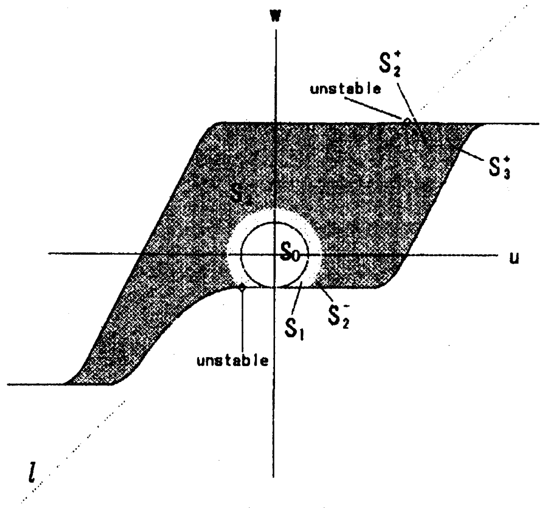

Fig. 16 $(u_0, w_0) \in S_3^-$

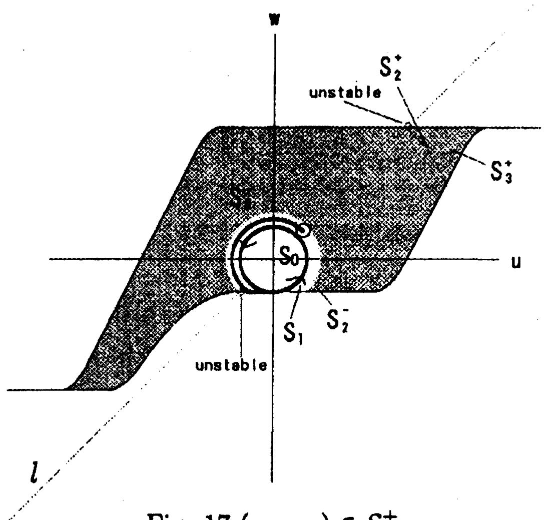

Fig. 17 $(u_0, w_0) \in S_3^+$

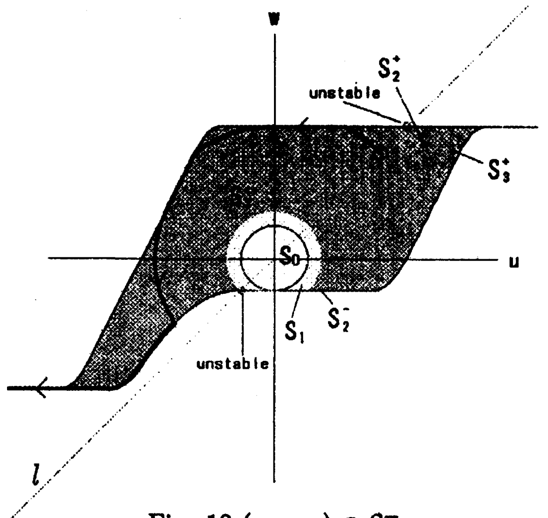

Fig. 18 $(u_0, w_0) \in S_3^-$

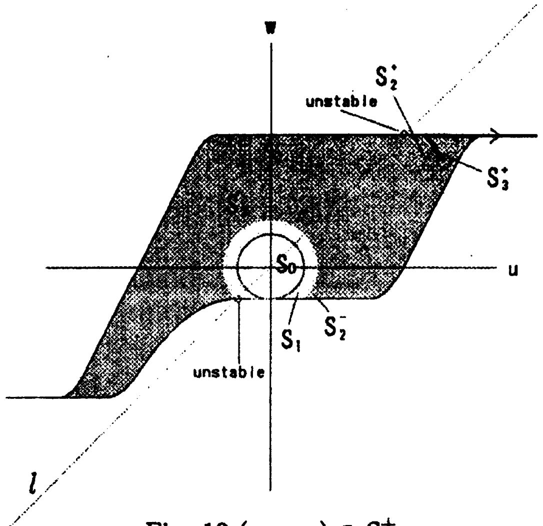

Fig. 19 $(u_0, w_0) \in S_3^+$

These numerical experiments show that the subsets $S_i (i = 0, 1, 2, 3)$ are completely different from those in experiment 3. When the initial datum is $(0.8, 0.8)$, the orbit draws an anticlockwise arc and a part of Γ^* alternately and reaches Γ_* in a finite time, and moving along the curve $w = f_*(u)$ downward and diverges to $(-\infty, -1)$ as $t \to +\infty$.

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